AN ANALOG TO DE MOIVRE'S THEOREM IN A PLANE POINT SYSTEM.

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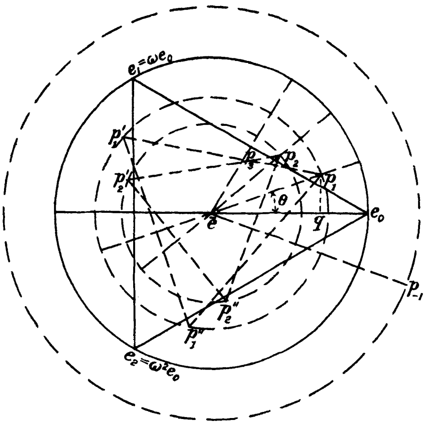
Let e_0, e_1, e_2 be three reference points for plane space, situated at the three vertices of an equilateral triangle, as in the figure; and let ω be an operator which changes e_0 into e_1, e_1 into e_2, and e_2 into e_0, i. e. rotates any point through 120°, about e, the centroid of the reference triangle. Then we shall have $\omega e_0 = e_1$, $\omega^2 e_0 = \omega e_1 = e_2$, $\omega^3 e_0 = \omega^2 e_1 = \omega e_2 = e_0$, etc., or $\omega^3 = 1$, $\omega^4 = \omega$, $\omega^5 = \omega^2$, $\omega^6 = 1$, etc.

Of course we may, if we choose, regard ω as $\frac{1}{2}(-1 + \sqrt{-3})$, one of the imaginary cube roots of unity, which possesses the above properties, but the work which follows does not depend upon its having this value.

It is proposed to investigate the operator $x_0 + x_1\omega + x_2\omega^2$ which transforms a point in a manner analogous to that in which $x + iy$ does a vector.

We write

$$(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 = x_0 e_0 + x_1 e_1 + x_2 e_2 = p_1 .$$



Now let us see what the effect will be of applying this operator again.

We have

$$
\begin{aligned}
&(x_0 + \omega x_1 + \omega^2 x_2)^2\, e_0 \\
&= (x_0 + \omega x_1 + \omega^2 x_2)\, p_1 \\
&= p_2 = (x_0 + \omega x_1 + \omega^2 x_2)(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 \\
&\qquad = x_0 (x_0 + \omega x_1 + \omega^2 x_2)\, e_0 + x_1 (\omega x_0 + \omega^2 x_1 + x_2)\, e_0 + x_2 (\omega^2 x_0 + x_1 + \omega x_2)\, e_0 \\
&\qquad = x_0 p_1 + x_1 p_1' + x_2 p_1'',
\end{aligned}
$$

say, in which p_1' is related to e_1, e_2, e_0 as p_1 is to e_0, e_1, e_2, and p_1'' is related to e_2, e_0, e_1 as p_1 is to e_0, e_1, e_2, i. e. $p_1' = \omega p_1$ and $p_1'' = \omega p_1' = \omega^2 p_1$. The location of p_1, p_1', p_1'' is shown in the figure.

Operating again, we have

$$(x_0 + \omega x_1 + \omega^2 x_2)\, p_2 = p_3 = (x_0 + \omega x_1 + \omega^2 x_2)\,(x_0 + \omega x_1 + \omega^2 x_2)\, p_1$$

$$= x_0\,(x_0 + \omega x_1 + \omega^2 x_2) p_1 + x_1\,(\omega x_0 + \omega^2 x_1 + x_2) p_1 + x_2\,(\omega^2 x_0 + x_1 + \omega x_2)\, p_1$$

$$= x_0 p_2 + x_1 p_2' + x_2 p_2'' = x_0 p_2 + x_1 \omega p_2 + x_2 \omega^2 p_2\,,$$

so that we have a new set of points p_2, p_2', p_2'', related to p_1, p_1', p_1'', as these are related to e_0, e_1, e_2. This process may evidently be repeated indefinitely.

Now these sets of points lie each on a circle whose center is at e, and the radii of these circles vary in length according to the following law. Let $T\bar{ee}_0 = a$, and $T\bar{ep}_1 = na$; then $T\bar{ep}_2 = n^2 a$, $T\bar{ep}_k = n^k a$. Again (angle between $\bar{ee}_0$ and $\bar{ep}_1$) $=$ (angle between $\bar{ep}_1$ and $\bar{ep}_2$) $=$ (angle between ep_2 and $\bar{ep}_3$), etc. Hence the operator $x_0 + \omega x_1 + \omega^2 x_2$ is a function of the angle θ between $\bar{ee}_0$ and $\bar{ep}_1$, and of the ratio n of the tensors $T\bar{ep}_1$ and $T\bar{ee}_0$; while $(x_0 + \omega x_1 + \omega^2 x_2)^k$ is a function of $k\theta$ and n^k. Again consider the point p_{-1} for which the angle is $-\,\theta$, and the distance from $\bar{e}$ is $\frac{1}{n}\,a$. The operator acting upon this point will evidently move it to e_0, that is $(x_0 + \omega x_1 + \omega^2 x_2) p_{-1} = e_0$, whence

$$p_{-1} = (x_0 + \omega x_1 + \omega^2 x_2)^{-1}\, e_0\,.$$

Evidently we shall have in the same way

$$p_{-k} = (x_0 + \omega x_1 + \omega^2 x_2)^{-k}\, e_0$$

in which, as well as in the expression above, k is to be taken as an integer. p_k is taken as a *unit* point, which requires that we have

$$x_0 + x_1 + x_2 = 1\,. \tag{1}$$

We will now determine n and θ as functions of the x's, and also $x_0 \ldots x_1$ as functions of n and θ.

We have for the length of the line joining two points $p_1 = \Sigma le$ and $p_2 = \Sigma me$,*

$$T(p_2 - p_1) = a\sqrt{3[(m_1 - l_1)^2 + (m_2 - l_2)^2 + (m_1 - l_1)(m_2 - l_2)}\,,$$

* See Hyde's Directional Calculus, Eq. 93.

which may, by symmetry, be also written

$$T(p_2 - p_1) = a \quad 3 \sqrt{(l_0 - m_0)^2 + (l_1 - m_1)^2 + (l_0 - m_0)(l_1 - m_1)}\,.$$

In this equation put (x_0, x_1, x_2) for (l_0, l_1, l_2), and let $m_0 = m_1 = m_2 = \frac{1}{3}$; then

$$Tep_1 = na = a\sqrt{3}\,\sqrt{(x_0 - \tfrac{1}{3})^2 + (x_1 - \tfrac{1}{3})^2 + (x_0 - \tfrac{1}{3})(x_1 - \tfrac{1}{3})}\,,$$

or

$$n = \sqrt{3}\,\sqrt{(x_0 - \tfrac{1}{3})^2 + (x_1 - \tfrac{1}{3})^2 + (x_0 - \tfrac{1}{3})(x_1 - \tfrac{1}{3})}\,. \tag{2}$$

Let us write $p_1 - e_0 = \rho$, $e_1 - e_0 = \varepsilon_1$, $e_2 - e_0 = \varepsilon_2$; $\therefore\ T(\varepsilon_1 + \varepsilon_2) = 3a$, and $p_1 - e = \rho - \frac{1}{3}(\varepsilon_1 + \varepsilon_2)$.

$$\operatorname{Cos}\theta = \frac{Teq}{Tep_1} = \frac{a - \rho \mid U(\varepsilon_1 + \varepsilon_2)}{na} = \frac{3a^2 - \rho \mid (\varepsilon_1 + \varepsilon_2)}{3na^2}\,.$$

But

$$\mid(\varepsilon_1 + \varepsilon_2) = 3a\,U(\varepsilon_2 - \varepsilon_1) = (\varepsilon_2 - \varepsilon_1)\sqrt{3}\,;$$

$$\cos\theta = \frac{3a^2 - \rho\,(\varepsilon_2 - \varepsilon_1)\sqrt{3}}{3na^2} = \frac{3a^2 - e_0 p_1 (e_2 - e_1)\sqrt{3}}{3na^2}$$

$$= \frac{3a^2 - \sqrt{3}\,(p_1 e_2 e_0 + p_1 e_0 e_1)}{3na^2} = \frac{3a^2 - \sqrt{3}\,(x_1 + x_2)}{3na^2}\,.$$

Now we always assume the double area of the reference triangle as unity,

$$e_0 e_1 e_2 = 1 = a\sqrt{3}\,.\,\tfrac{3}{2}a = \tfrac{3}{2}a^2\sqrt{3}\,,$$

whence

$$a^2 = \frac{2}{3\sqrt{3}}\,,$$

therefore

$$\cos\theta = \frac{2 - 3(x_1 + x_2)}{2n}\,. \tag{3}$$

From (3) and (1) we have

$$x_1 + x_2 = 1 - x_0 = \tfrac{2}{3}(1 - n\cos\theta)\,,$$

whence

$$x_0 = \tfrac{1}{3}(1 + 2n\cos\theta)\,. \tag{4}$$

Substituting this value of x_0 in eq. (2) we have

$$n^2 = 3\,[\tfrac{4}{9}n^2\cos^2\theta + (x_1 - \tfrac{1}{3})^2 + \tfrac{2}{3}n\cos\theta\,(x_1 - \tfrac{1}{3})]\,,$$

from which we obtain

$$x_1 = \tfrac{1}{3}(1 - n\cos\theta + n\sqrt{3}\sin\theta) = \tfrac{1}{3}\left[1 - 2n\cos\left(\theta + \frac{\pi}{3}\right)\right], \tag{5}$$

and by (1) and (4)

$$x_2 = \tfrac{1}{3}(1 - n\cos\theta - n\sqrt{3}\sin\theta) = \tfrac{1}{3}\left[1 - 2n\cos\left[\theta - \frac{\pi}{3}\right]\right]. \tag{6}$$

We have thus found the x's as functions of n and θ: let us call them K functions; thus

$$\left.\begin{aligned} K_0(n, \theta) &= \tfrac{1}{3}(1 + 2n\cos\theta) \\ K_1(n, \theta) &= \tfrac{1}{3}\left[1 - 2n\cos\left[\theta + \frac{\pi}{3}\right]\right] \\ K_2(n, \theta) &= \tfrac{1}{3}\left[1 - 2n\cos\left[\theta - \frac{\pi}{3}\right]\right] \end{aligned}\right\}, \tag{7}$$

and we have

$$K_0(n, \theta) + K_1(n, \theta) + K_2(n, \theta) = 1, \tag{8}$$

an equation analogous to $\cos^2\theta + \sin^2\theta = 1$.

Using K's now for x's we may write

$$\begin{aligned}(K_0(n, \theta) + \omega K_1(n, \theta) + \omega^2 K_2(n, \theta))^k \\ = K_0(n^k, k\theta) + \omega K_1(n^k, k\theta) + \omega^2 K_2(n^k, k\theta),\end{aligned} \tag{9}$$

which is the analog of De Moivre's theorem. We have shown it to be true for all integral values of k, and will now show it to hold for fractional values. Let us write for brevity

$$F(n, \theta) = K_0(n, \theta) + \omega K_1(n, \theta) + \omega^2 K_2(n, \theta), \tag{10}$$

then, since, when k is an integer, we have

$$[F(n, \theta)]^k = F(n^k, k\theta),$$

therefore $F(n, \theta)$ is one value of

$$[F(n^k, k\theta)]^{1/k}.$$

But

$$[F(n, \theta)]^{l/k} = \{[F(n, \theta)]^l\}^{1/k} = [F(n^l, l\theta)]^{1/k},$$

of which, as above, one value is

$$F\left[n^{l/k}, \frac{l}{k}\theta\right],$$

and hence the theorem holds for all real values of k.

To find the k values of the kth root we have, as in De Moivre's Theorem,

$$F(n, \theta) \equiv F(n, \theta + 2r\pi),$$

$$[F(n, \theta)]^{1/k} \equiv [F(n, \theta + 2r\pi)]^{1/k} \equiv F\left[n^{1/k}, \frac{\theta + 2r\pi}{k}\right]. \tag{11}$$

Addition-multiplication Theorem for the K-functions.

Let

$$p_1 = (x_0 + \omega x_1 + \omega^2 x_2)\, e_0, \quad q_1 = (y_0 + \omega y_1 + \omega^2 y_2)\, e_0,$$

and

$$p' = (x_0 + \omega x_1 + \omega^2 x_2)\, q_1 = (x_0 + \omega x_1 + \omega^2 x_2)(y_0 + \omega y_1 + \omega^2 y_2)\, e_0$$
$$= (y_0 + \omega y_1 + \omega^2 y_2)(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 = (y_0 + \omega y_1 + \omega^2 y_2)\, p_1.$$

These equations show that p' is obtained by performing the x operation on q_1, or the y operation on p_1. The result is evidently to turn ee_0 through the angle $\theta_1 + \theta_2$ and to multiply its length by $n_1 n_2$; then expanding the value of p', and putting $K_0(n_1, \theta_1)$ for x_0, $K_0(n_2, \theta_2)$ for y_0, etc., we have

$$p' = [(x_0 y_0 + x_1 y_2 + x_2 y_1) + (x_2 y_2 + x_1 y_0 + x_0 y_1)\,\omega + (x_1 y_1 + x_2 y_0 + x_0 y_2)\,\omega^2]\, e_0$$
$$= [K_0(n_1 n_2, \theta_1 + \theta_2) + \omega K_1(n_1 n_2, \theta_1 + \theta_2) + \omega^2 K_2(n_1 n_2, \theta_1 + \theta_2)]\, e_0.$$

$$\left.\begin{aligned}
K_0(n_1 n_2, \theta_1 + \theta_2) &= K_0(n_1, \theta_1) K_0(n_2, \theta_2) + K_1(n_1, \theta_1) K_2(n_2, \theta_2) \\
&\qquad + K_2(n_1, \theta_1) K_1(n_2, \theta_2) \\
K_1(n_1 n_2, \theta_1 + \theta_2) &= K_2(n_1, \theta_1) K_2(n_2, \theta_2) + K_0(n_1, \theta_1) K_1(n_2, \theta_2) \\
&\qquad + K_1(n_1, \theta_1) K_0(n_2, \theta_2) \\
K_2(n_1 n_2, \theta_1 + \theta_2) &= K_1(n\;, \theta_1) K_1(n_2, \theta_2) + K_2(n_1, \theta_1) K_0(n_2, \theta_2) \\
&\qquad + K_0(n_1, \theta_1) K_2(n_2, \theta_2)
\end{aligned}\right\}. \tag{12}$$

If in these equations we make $n_1 = n_2 = n$, and $\theta_1 = \theta_2 = \theta$, we have

$$\left.\begin{aligned}
K_0(n^2, 2\theta) &= [K_0(n, \theta)]^2 + 2K_1(n, \theta) K_2(n, \theta) \\
K_1(n^2, 2\theta) &= [K_2(n, \theta)]^2 + 2K_0(n, \theta) K_1(n, \theta) \\
K_2(n^2, 2\theta) &= [K_1(n, \theta)]^2 + 2K_2(n, \theta) K_0(n, \theta)
\end{aligned}\right\}. \tag{13}$$

Some special values of the K functions. By (7)

$$\left.\begin{aligned}
K_0(n, -\theta) &= K_0(n, \theta) \\
K_1(n, -\theta) &= K_2(n, \theta) \\
K_2(n, -\theta) &= K_1(n, \theta)
\end{aligned}\right\}. \tag{14}$$

$$\left.\begin{aligned} K_0(n,0) &= \tfrac{1}{3}(1+2n)\,, & K_0(1,0) &= 1\,, \\ K_1(n,0) &= \tfrac{1}{3}(1-n)\,, & K_1(1,0) &= 0\,, \\ K_2(n,0) &= \tfrac{1}{3}(1-n)\,, & K_2(1,0) &= 0\,, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} K_0\left[-1,\frac{\pi}{3}\right] &= 0\,, & K_0\left[1,\frac{2\pi}{3}\right] &= 0\,, \\ K_1\left[-1,\frac{\pi}{3}\right] &= 0\,, & K_1\left[1,\frac{2\pi}{3}\right] &= 1\,, \\ K_2\left[-1,\frac{\pi}{3}\right] &= 1\,, & K_2\left[1,\frac{2\pi}{3}\right] &= 0\,. \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} K_0\left[n,\frac{\pi}{2}\right] = K_1\left[n,\frac{\pi}{6}\right] = K_2\left[n,\frac{5\pi}{6}\right] &= \tfrac{1}{3} \\ K_1\left[n,\frac{\pi}{2}\right] = K_0\left[n,\frac{\pi}{6}\right] = K_1\left[n,\frac{5\pi}{6}\right] &= \tfrac{1}{3}(1+n\sqrt{3}) \\ K_2\left[n,\frac{\pi}{2}\right] = K_2\left[n,\frac{\pi}{6}\right] = K_0\left[n,\frac{5\pi}{6}\right] &= \tfrac{1}{3}(1-n\sqrt{3}) \end{aligned}\right\}. \tag{17}$$

By (10) and (16)

$$\omega^k = \left[F\left[1,\frac{2\pi}{3}\right]\right]^k = F\left[1,\frac{2k\pi}{3}\right], \tag{18}$$

so that ω^k is an operator that moves e_0 in the circumscribing circle through k times the arc of 120°. Hence the equation of this circle may be written

$$p = \omega^x e_0\,. \tag{19}$$

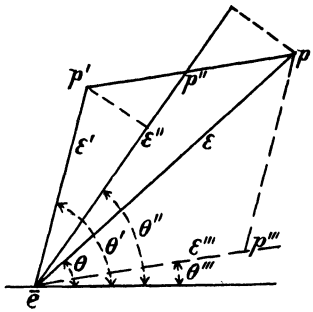
We may also write the equation of a logarithmic spiral through e_0, as

$$p = F(n^x, x\theta)\,. \tag{20}$$

We have also the following relations

$$\begin{aligned} K_0(n,\theta) &= K_1\left[n,\theta+\frac{2\pi}{3}\right] = K_2\left[n,\theta-\frac{2\pi}{3}\right] \\ K_1(n,\theta) &= K_2\left[n,\theta+\frac{2\pi}{3}\right] = K_0\left[n,\theta-\frac{2\pi}{3}\right] \\ K_2(n,\theta) &= K_0\left[n,\theta+\frac{2\pi}{3}\right] = K_1\left[n,\theta-\frac{2\pi}{3}\right] \end{aligned}$$

Sums and Differences of the K Functions.



Let

$$p = \sum_0^2 xe\,, \quad p' = \sum_0^2 x'e\,;$$

$$p'' = \Sigma \left[\frac{x + x'}{2} \,.\, e \right]$$

is the bisecting point of pp'.

Let $Tep = na$, $Tep' = n'a$, $T\bar{e}p'' = n''a$, and $p - \bar{e} = \varepsilon$, $p' - \bar{e} = \varepsilon'$, $p'' - e = \varepsilon''$; then

$$T^2ep'' = n''^2a^2 = \tfrac{1}{4}\,(\varepsilon + \varepsilon')^{\underline{2}} = \tfrac{1}{4}\,(\varepsilon^{\underline{2}} + \varepsilon'^{\underline{2}} + 2\varepsilon \mid \varepsilon')$$

$$= \frac{a^2}{4}\,(n^2 + n'^2 + 2nn' \cos(\theta - \theta')\,)$$

$$\therefore \qquad n'' = \tfrac{1}{2}\sqrt{n^2 + n'^2 + 2nn' \cos(\theta - \theta')}\,. \tag{21}$$

Again

$$an \sin(\theta'' - \theta) = an' \sin(\theta' - \theta'')\,,$$

or

$$n\,(\sin\theta'' \cos\theta - \cos\theta'' \sin\theta) = n'\,(\sin\theta' \cos\theta'' - \cos\theta' \sin\theta'')\,,$$

whence

$$\tan\theta'' = \frac{n \sin\theta + n' \sin\theta'}{n \cos\theta + n' \cos\theta'}\,. \tag{22}$$

If $n' = n$, (21) and (22) become

$$n'' = n \cos\frac{\theta - \theta'}{2} \quad \text{and} \quad \theta'' = \frac{\theta + \theta'}{2}\,, \tag{23}$$

while, if $\theta' = \theta$,

$$n'' = \tfrac{1}{2}\,(n + n'), \quad \text{and} \quad \theta'' = \theta\,. \tag{24}$$

We have therefore in general

$$K(n, \theta) + K(n', \theta')$$

$$= 2K\left[\tfrac{1}{2}\sqrt{n^2 + n'^2 + 2nn' \cos(\theta - \theta')}\,,\ \tan^{-1}\frac{n \sin\theta + n' \sin\theta'}{n\cos\theta + n'\cos\theta'}\right]. \quad (25)$$

When $n' = n$

$$K(n, \theta) + K(n, \theta') = 2K\left[n \cos\frac{\theta - \theta'}{2}\,,\ \frac{\theta + \theta'}{2}\right], \quad (26)$$

and when $\theta' = \theta$

$$K(n, \theta) + K(n', \theta) = 2K\left[\frac{n + n'}{2}\,,\ \theta\right]. \quad (27)$$

Subtracting we have

$$p - p' = \varepsilon''' = p''' - e = \Sigma\,(x - x')\,e\,;$$

so that

$$p''' = \Sigma K(n''', \theta''')\,e = \Sigma\,(x - x')\,e + \tfrac{1}{3}\Sigma e = \Sigma\,(x - x' + \tfrac{1}{3})\,e$$

$$= \Sigma\,[(K(n, \theta) - K(n', \theta') + \tfrac{1}{3})\,e]\,;$$

$$\therefore \quad K(n, \theta) - K(n', \theta') = K(n''', \theta''') - \tfrac{1}{3}\,. \quad (28)$$

But

$$T\varepsilon''' = T(\varepsilon - \varepsilon') = \sqrt{\varepsilon^2 + \varepsilon'^2 - 2\varepsilon \mid \varepsilon'},$$

$$\therefore \quad n''' = \sqrt{n^2 + n'^2 - 2nn' \cos(\theta - \theta')}$$

and

$$\theta''' = \tan^{-1}\frac{n \sin\theta - n' \sin\theta'}{n\cos\theta - n'\cos\theta'},$$

hence

$$K(n, \theta) - K(n', \theta')$$

$$= K\left[\sqrt{n^2 + n'^2 - 2nn' \cos(\theta - \theta')}\,,\ \tan^{-1}\frac{n \sin\theta - n' \sin\theta'}{n\cos\theta - n'\cos\theta'}\right] - \tfrac{1}{3}\,. \quad (29)$$

When $n' = n$

$$K(n, \theta) - K(n, \theta') = K\left[2n \sin\frac{\theta - \theta'}{2}\,,\ \frac{\theta + \theta' + \pi}{2}\right] - \tfrac{1}{3}\,. \quad (30)$$

When $\theta' = \theta$

$$K(n, \theta) - K(n', \theta) = K[n - n', \theta] - \tfrac{1}{3}\,. \quad (31)$$

We have thus a trigonometry of the K functions analogous to that of the circular functions.

Besides whatever interest these relations may have due to their analogies, they might prove of use when working with a point system in plane space, for comparing the location of points by weights with that by angle and distance.